

Using the subspace theorem to bound unit distances

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Abstract

We prove a special case of Erdős' unit distance problem using a corollary of the subspace theorem bounding the number of solutions of linear equations from a multiplicative group. We restrict our attention to unit distances coming from a multiplicative group of rank r not too large. Specifically, given $\varepsilon > 0$ and n points in the plane, we construct the unit distance graph from these points and distances and use the corollary above to bound certain paths of length k in the graph giving at most $n^{1+\varepsilon}$ unit distances from the group above. We require that the rank $r \leq c \log n$ for some $c > 0$ depending on ε . This extends a result of József Solymosi, Frank de Zeeuw and the author where we only considered unit distances that are roots of unity. Lastly we show that the lower bound configuration for the unit distance problem of Erdős consists of unit distances from a multiplicative subgroup of the form above.

1 Introduction

In 1946 Erdős asked for the maximal possible number, $u(n)$, of unit distances among n points in the plane. He gave the lower bound $u(n) > n^{1+c/\log \log n}$, using a $\sqrt{n} \times \sqrt{n}$ grid and conjectured that this was the true magnitude [5]. The best known upper bound is $u(n) < cn^{4/3}$, first proved by Spencer, Szemerédi and Trotter in 1984 [16]. This bound has many proofs, the simplest of which was the proof by Székely [17], using the crossing inequality for graphs. As a general reference for work done on the unit distance problem see [3].

In [15] József Solymosi, Frank de Zeeuw and the author showed that the number of unit distances between n points in the plane with angle to the x -axis a rational multiple of π is at most $n^{1+c/\sqrt{\log n}}$. Such unit distances correspond to roots of unity. The bound was found by using a result of H.B. Mann bounding sums of roots of unity [10]. This work is an extension of that result to unit distances from a group of finite rank—roots of unity correspond to rank 0. The proof follows in almost the same way except we use the subspace theorem instead of Mann's result.

Our main tool is a corollary of the subspace theorem. The subspace theorem was first proved by W.M. Schmidt in 1972 [14]. This theorem essentially says

that solutions of linear equations, in a multiplicative subgroup of a field of finite rank, come from a finite number of linear subspaces. A number of improvements, including quantitative versions, of this result have been made. The corollary we use bounds the number of such solutions depending on the rank and dimension. This corollary was due originally to Evertse, Schlickewei and Schmidt [7]. We will use an improvement of this result of Amoroso and Viada [1]. The progression of these results is given in [7, 6]. We will use the bound from [1] which is, to our knowledge, the best known bound for the corollary. The subspace theorem is a very powerful result with a wealth of applications in number theory. For some examples see [2].

Consider two points $p, q \in \mathbb{R}^2$ with unit distance. Considering the vector between these two points we get the complex number $z = z(p, q) = \overrightarrow{pq}$ with $|z| = 1$. We will restrict our attention to unit distances with z coming from a multiplicative subgroup of \mathbb{C}^* (the multiplicative group of nonzero complex numbers) of finite rank. A subgroup $\Gamma \subset \mathbb{C}^*$ has rank r if there exists a finitely generated subgroup $\Gamma_0 \subset \Gamma$ with r generators such that for every $x \in \Gamma$ there exists an integer $k \geq 0$ such that $x^k \in \Gamma_0$.

Suppose Γ is a subgroup of \mathbb{C}^* of finite rank r and $a_1, a_2, \dots, a_k \in \mathbb{C}^*$. A solution of the equation $a_1 z_1 + a_2 z_2 + \dots + a_k z_k = 1$ is called *nondegenerate* if no subsum of the left hand side vanishes. That is $\sum_{j \in J} a_j z_j \neq 0$ for every nontrivial $J \subset \{1, 2, \dots, k\}$. We will consider the number $A(k, r)$ of nondegenerate solutions of this equation with $z_i \in \Gamma$. We now give the corollary of the subspace theorem that we need.

Theorem 1. *Suppose $a_1, a_2, \dots, a_k \in \mathbb{C}^*$ and Γ has finite rank r . Then the number of nondegenerate solutions of the equation*

$$a_1 z_1 + a_2 z_2 + \dots + a_k z_k = 1 \tag{1}$$

with $z_i \in \Gamma$ is at most

$$A(k, r) \leq (8k)^{4k^4(k+kr+1)}.$$

Theorem 1 was proved over an arbitrary algebraically closed field K of characteristic 0 but we only require it over \mathbb{C} .

We will use this to prove the following result.

Theorem 2. *Let $\varepsilon > 0$. Then there exist $n_0 = n_0(\varepsilon)$ a positive integer and $c = c(\varepsilon) > 0$ such that given $n > n_0$ points in the plane, the number of unit distances with z coming from a subgroup $\Gamma \subset \mathbb{C}^*$ with rank $r < c \log n$ is at most $n^{1+\varepsilon}$.*

We will prove this theorem in the next section and in Section 3 we will show that the lower bound of Erdős satisfies the hypotheses of this theorem.

2 Proof of the main result

The proof of Theorem 2 is quite similar to the proof of Theorem 2.2 in [15]. The main difference is that we use the subspace theorem instead of Mann's result to get an upper bound for paths in the unit distance graph.

Suppose $G = G(V, E)$ is a graph on $v(G) = n$ vertices and $e(G) = cn^{1+\alpha}$ edges. We denote the minimum degree in G by $\delta(G)$. The following lemma shows that we can remove low degree vertices from our graph without greatly affecting the number of edges.

Lemma 3. *Let G be as above. Then G contains a subgraph H with $e(H) = (c/2)n^{1+\alpha}$ edges such that $\delta(H) \geq (c/2)n^\alpha$.*

Proof. We remove vertices from G of degree less than $(c/2)n^\alpha$. Then, the resulting subgraph H has $\delta(H) \geq (c/2)n^\alpha$ and we removed fewer than $(c/2)n^{1+\alpha}$ edges so H contains more than $(c/2)n^{1+\alpha}$ edges. \square

Note that the subgraph H constructed above contains at least $v(H) = \sqrt{cn}^{1/2+\alpha/2}$ vertices.

Suppose we are given a path on k edges $P_k = p_0 p_1 \dots p_k$. We call this path *irredundant* if

$$\sum_{i \in I} \overrightarrow{p_i p_{i+1}} \neq 0$$

for every $\emptyset \neq I \subset \{0, 1, \dots, k-1\}$.

Proof of Theorem 2. Let G be the graph with the n points in the plane as vertices and the unit distances with z coming from Γ as edges. Suppose there are $n^{1+\varepsilon}$ such distances. Then $e(G) = n^{1+\varepsilon}$. We will show that we can take ε as small as we like. We will count the number of irredundant paths P_k in G , for a fixed k that we will choose later. By Lemma 3 we can assume that $e(G) \geq (1/2)n^{1+\varepsilon}$, $v(G) \geq n^{1/2+\varepsilon/2}$ and $\delta(G) \geq (1/2)n^\varepsilon$.

The number of irredundant paths P_k starting at any vertex v is at least

$$N \geq \prod_{\ell=0}^{k-1} (\delta(G) - 2^\ell + 1) \geq \frac{n^{k\varepsilon}}{2^{2k}}.$$

The first inequality is true since if we have constructed a subpath P_ℓ of P_k , then at most $2^\ell - 1$ of the at least $\delta(G)$ possible continuations are forbidden. In the second inequality we have assumed that $2^k \leq (1/2)n^\varepsilon$, which is true as long as $k < \varepsilon \log n / \log 2 - 1$ (we will show that this holds at the end of the proof). Thus the total number of irredundant paths P_k is at least $N n^{1/2+\varepsilon/2} / 2 \geq n^{1/2+(k+1/2)\varepsilon} / 2^{2k+1}$. It follows that there are two vertices v and w with at least $N n^{1/2+\varepsilon/2} / n^2 \geq n^{(k+1/2)\varepsilon-3/2} / 4^k$ irredundant paths P_k between them. We will call the set of these paths \mathcal{P}_{vw} , so that we have

$$|\mathcal{P}_{vw}| \geq \frac{n^{(k+1/2)\varepsilon-3/2}}{4^k}.$$

Given $P_k \in \mathcal{P}_{vw}$, $P_k = p_0 p_1 \dots p_k$, consider the k -tuple (z_1, \dots, z_k) where z_i is the complex number in the direction from p_{i-1} to p_i , i.e. $z_i = z(p_{i-1}, p_i) = \overrightarrow{p_{i-1} p_i}$. Let $a = z(v, w)$. Then $z_1 + z_2 + \dots + z_k = a$. Since the path is irredundant

no subsum on the left vanishes. So P_k corresponds to a nondegenerate solution of Equation (1) with $a_i = 1/a$ for $i = 1, 2, \dots, k$. Thus, by Theorem 1,

$$|\mathcal{P}_{vw}| \leq (8k)^{4k^4(k+kr+1)}.$$

Putting these inequalities together and taking logarithms we get

$$\begin{aligned} ((k+1/2)\varepsilon - 3/2) \log n &\leq k \log 4 + 4k^4(k+kr+1) \log(8k) \\ &\leq 5rk^5 \log k, \end{aligned}$$

where the last inequality holds for large k . From this we get

$$\varepsilon \leq \frac{5rk^5 \log k}{(k+1/2) \log n} + \frac{3}{2(k+1/2)} \leq \frac{5rk^4 \log k}{\log n} + \frac{3}{2k}. \quad (2)$$

We consider the expression on the right hand side as a function of k . Optimizing this function we get

$$k \geq \exp\left((1/5)W(5c_2 \log n/r)\right)$$

for some constant $c_2 > 0$ where W is the positive real-valued function satisfying $x = W(x)e^{W(x)}$. This function is called the Lambert W function and was first studied by J.H. Lambert in 1758 [9]. The following asymptotic expression is due to N.G. de Bruijn [4]:

$$W(x) = \log(x) - \log \log(x) + O\left(\frac{\log \log \log x}{\log \log x}\right).$$

We don't require this much accuracy. One can easily check, and we will just use the fact, that $(1/2) \log x \leq W(x) \leq \log x$ for $x \geq e$.

Then we can take

$$c' \left(\frac{\log n}{r}\right)^{1/5} \leq k \leq c'' \left(\frac{\log n}{r}\right)^{1/5}$$

for some constants $c', c'' > 0$.

For any $\varepsilon > 0$ there is a constant $c > 0$ such that if $r+1 \leq c \log n$ then the inequality in (2) holds for large n . When counting P_k 's we made the assumption that $k \leq \varepsilon \log n / \log 2 - 1$. Checking the above values of k and f we see that this holds for large n .

This completes the proof. \square

3 Analysis of Erdős' lower bound

It would be interesting to analyze the possible group structure of unit distances from a maximal set of points. We will now show that the lower bound configuration for the unit distance problem given by Erdős satisfies the hypotheses of

Theorem 2. Matoušek has given a very in-depth account of Erdős' lower bound and we will follow that here [11].

We require the following number theoretic functions:

$$\pi_{d,a}(x) = \sum_{\substack{p \leq x \\ p \equiv a(d)}} 1, \quad \vartheta_{d,a}(x) = \sum_{\substack{p \leq x \\ p \equiv a(d)}} \log p, \quad \psi_{d,a}(x) = \sum_{\substack{p^\ell \leq x \\ p^\ell \equiv a(d)}} \log p,$$

where the first two sums are over primes less than x of the form $p = a + kd$ and the last sum is over primes p and positive integers ℓ such that $p^\ell = a + kd$ and $p^\ell \leq x$. These are analogues of the prime counting function and Chebyshev functions for arithmetic progressions.

We will use the following results regarding these functions all of which are well known in number theory. For details see [8] and [12].

Theorem 4 (The Prime Number Theorem for Arithmetic Progressions). *Suppose a and d are positive integers such that $(a, d) = 1$. Then*

$$\pi_{d,a}(n) = (1 + o(1)) \frac{1}{\varphi(d)} \cdot \frac{n}{\log n}.$$

A simple consequence of this result is that if a and d are positive integers such that $a < d$ and $(a, d) = 1$ then the k th prime of the form $p_i = a + k_i d$ satisfies $p_k = (1 + o(1))k \log k / \varphi(d)$.

Theorem 5. *Suppose a and d are positive integers such that $(a, d) = 1$. Then*

$$\psi_{d,a}(n) = (1 + o(1)) \frac{n}{\varphi(d)}.$$

Theorem 5 can be deduced from Theorem 4 by partial summation.

Theorem 6. *Suppose a and d are positive integers such that $(a, d) = 1$. Then*

$$\vartheta_{d,a}(n) = (1 + o(1)) \psi_{d,a}(n).$$

The above two theorems give $\vartheta_{d,a}(n) = (1 + o(1))n / \varphi(d)$.

We will also use the following fact. For details see [13].

Theorem 7. *The number of integer solutions, $R(m)$, of $x^2 + y^2 = m$ where $m = p_1 p_2 \dots p_r$ and the p_i are distinct primes of the form $p_i = 4k_i + 1$ is*

$$R(m) = 2^{r+2}.$$

The lower bound configuration consists of n points in a $\sqrt{n} \times \sqrt{n}$ grid. The step in the grid is chosen to be $1/\sqrt{m}$ where m is the product of the first $r - 1$ primes of the form $4k + 1$ and r is the largest number with $m \leq n/4$. We will in fact consider a $\sqrt{n} \times \sqrt{n}$ grid with step 1 and then count the distances of length \sqrt{m} . This gives a lower bound to the unit distance problem by scaling the point set by $1/\sqrt{m}$.

We have $4p_1p_2\ldots p_{r-1} \leq n < 4p_1p_2\ldots p_r$. From this the bound $r \geq \log n / (3 \log \log n)$ is found using the prime number theorem for arithmetic progressions. Distances equal to \sqrt{m} in this configuration correspond to integer solutions of $x^2 + y^2 = m$. In the lower bound, the fact that there are at least $2^{r-1}/16$ such distances is used. But an upper bound on the number of such distances can also be found. By Theorem 7 there are at most $4.2^{(r-1)+2} = 2^{r+3}$ such distances from any point so we have at most $2^{r+3}n$ such distances in total. Erdős' construction gives a lower bound for r . If we can find an upper bound for r then we are done as will be described below.

We will briefly explain the reason that m is defined as above as this highlights the generators to choose for a multiplicative subgroup of \mathbb{C}^* containing the unit distances of the configuration. A prime p has a unique expression, up to the order of the terms, of the form $x^2 + y^2 = p$ with x and y positive integers if and only if $p = 2$ or $p = 4k + 1$ for some integer k . The Brahmagupta-Fibonacci identity says that the product of two numbers, each expressible as the sum of two squares, is itself expressible as the sum of two squares. Specifically

$$\begin{aligned} (a^2 + b^2)(c^2 + d^2) &= (ac - bd)^2 + (ad + bc)^2 \\ &= (ac + bd)^2 + (ad - bc)^2. \end{aligned}$$

So as we multiply more primes of the form $4k + 1$ together we get more expressions of the resulting number as a sum of two squares. So all solutions of $x^2 + y^2 = m$ can be described in terms of the solutions of $x_j^2 + y_j^2 = p_j$.

More formally, we consider the ring R of points in \mathbb{Z}^2 with addition defined coordinate-wise, so $(a, b) + (c, d) = (a + c, b + d)$, and multiplication defined as follows $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. One can check that R is actually a ring and is in fact isomorphic to the Gaussian integers $\mathbb{Z}[i]$ since the operations correspond to complex addition and complex multiplication. But $\mathbb{Z}[i]$ is a unique factorization domain so R is also a unique factorization domain. We will consider the elements of R as distance vectors.

In our grid we are looking for the distance $m = p_1 \ldots p_{r-1}$. By Theorem 7, the number of pairs $(x, y) \in \mathbb{Z}^2$ with $x^2 + y^2 = m$ is $R(m) = 2^{r+1}$. Suppose $x_j^2 + y_j^2 = p_j$. We consider the point $(x_j, y_j) \in R$. The product of these $r - 1$ points $(x, y) = (x_1, y_1)(x_2, y_2) \ldots (x_{r-1}, y_{r-1})$ has magnitude

$$|(x, y)| = |(x_1, y_1)| \ldots |(x_{r-1}, y_{r-1})| = \sqrt{p_1 \ldots p_{r-1}} = \sqrt{m}.$$

So this product gives a point with length \sqrt{m} .

Now, R is a unique factorization domain. That means that the point $(x, y) = (x_1, y_1) \ldots (x_{r-1}, y_{r-1})$ has unique factorization. Specifically, in any other factorization of $(x, y) = (x'_1, y'_1) \ldots (x'_{r-1}, y'_{r-1})$ there is a bijection ϕ of the factors such that $(x_j, y_j) = u_j(x'_{\phi(j)}, y'_{\phi(j)})$ where u_j is a unit. The units in R correspond to the units in $\mathbb{Z}[i]$. In the latter these are $1, -1, i, -i$ so in the former they are $(1, 0), (-1, 0), (0, 1)$ and $(0, -1)$. Two elements $(a, b), (c, d) \in R$ are called *associates* if $(a, b) = u(c, d)$ for some unit u . So in a unique factorization domain the factorization of an element is unique up to ordering and

associates. Now, since the p_j 's are odd primes we cannot have $x_j = \pm y_j$ for $1 \leq j \leq r-1$. One can check that (x_j, y_j) and $(x_j, -y_j)$ are not associates and $(x_j, y_j), (x_k, y_k)$ are not associates for $j \neq k$. So we have two points to choose from for each p_j , namely (x_j, y_j) and $(x_j, -y_j)$, giving 2^{r-1} choices for (x, y) . None of the factors are associates so these choices for (x, y) are all distinct. If we multiply a given (x, y) by a unit then we get four different values. So we get $4 \cdot 2^{r-1} = 2^{r+1}$ distinct points (x, y) each with length \sqrt{m} . So these give all possible required distances by Theorem 7. The units are torsion points of R (they have finite multiplicative order in R) so they don't affect the rank.

Going back to unit distances, if we take the complex numbers

$$z_j = m^{-1/(2r-2)}(x_j + iy_j), \quad w_j = m^{-1/(2r-2)}(x_j - iy_j)$$

for $1 \leq j \leq r-1$ then these generate the multiplicative group of unit distances in the configuration. Thus the unit distances come from a multiplicative subgroup of \mathbb{C}^* of rank at most $r-1$. So we just need to bound r from above.

We do this by looking at the inequality $p_1 \dots p_{r-1} \leq n/4$. Taking logarithms we get $\vartheta_{4,1}(p_{r-1}) \leq \log(n/4)$. By Theorems 5 and 6 we get

$$\frac{p_{r-1}}{2\sqrt{2}} \leq \log(n/4).$$

By the remark after Theorem 4 we get

$$\frac{(r-1) \log(r-1)}{2\sqrt{2}} \leq 2\sqrt{2} \log(n/4).$$

Solving for r we get

$$r \leq \frac{16 \log n}{\log \log n}.$$

Thus the unit distances come from a multiplicative subgroup of rank at most $r-1 \leq 16 \log n / \log \log n - 1$. Since

$$\frac{16 \log n}{\log \log n} - 1 \leq c \log n$$

for large n this configuration is covered by Theorem 2.

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